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Canards and critical behavior in autocatalytic combustion models

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Abstract The basic elements of the theory of slow invariant manifolds and canard phenomena of singularly perturbed nonlinear differential equations in the context of thermal-explosion problems are outlined. The mathematical results are applied to the investigation of the critical phenomena in autocatalytic combustion models described by singularly perturbed differential equations with lumped and distributed parameters. Critical regimes are modeled by canards (one-dimensional stable-unstable slow invariant manifolds). The geometric approach in combination with asymptotic and numeric methods permits to explain the strong parametric sensitivity and to obtain asymptotic representations of the critical conditions of self-ignition.

Keywords Canards · Combustion · Invariant manifolds · Singular perturbations · Thermal explosion

1 Introduction

The evaluation of critical regimes thought of as regimes separating the regions of explosive and nonexplosive chemical reactions is the main mathematical problem of thermal-explosion theory. The interest in critical phenomena is brought about not only for reasons of safety; indeed, the critical regime is often the most effective in technological processes. Here the sense of criticality is as follows. The critical regime

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corresponds to a chemical reaction separating the domains of self-accelerating reactions and domains of slow reactions.

Various investigations of critical phenomena in thermal-explosion theory have been reported in [1–7]. Because of the considerable difference between velocities related to thermal and concentrational changes, singularly perturbed systems of differential equations serve as mathematical models of such problems. But in the above works the authors restrict their consideration to the study of the zero-order approximation. This does not permit them to explain the strong parametric sensitivity of this problem, as well as to examine the transformation of the solutions in the vicinity of the limit of self-ignition.

An important part of the paper is dedicated to modeling critical combustion regimes and to finding critical values of the control parameters using novel mathematical methods based on the theory of "canards" [8]. In the majority of the literature devoted to canards the term "canard" is associated with periodic trajectories [9]. In our work *a canard* is a trajectory of a singularly perturbed system of differential equations if it follows first a stable invariant manifold, and then an unstable one. In both cases the distances traveled are not infinitesimally small. It should be noted that a canard may be considered as a result of gluing stable (attractive) and unstable (repelling) slow invariant manifolds at one point of the breakdown surface due to the availability of an additional scalar parameter in the differential system. We shall use canards as *separating solutions* corresponding to the critical regimes of chemical reactions. This approach was proposed for the first time in [10, 11] and was then applied in [12–14]. This approach permits to work out the algorithms of asymptotic representations of the critical values of the parameter of initial conditions and to describe the transfer regimes.

2 Statement of the problem

2.1 Singular perturbations and canards

The main object of our consideration is the following singularly perturbed system

$$\frac{dx}{dt} = f(x, y, \varepsilon), \tag{1}$$

$$\varepsilon\frac{dy}{dt} = g(x, y, \alpha, \varepsilon), \tag{2}$$

where ε is a small positive parameter, α is a scalar parameter, y is a scalar variable, x is a vector of dimension n. The case of the vector variable y can be considered as well.

Recall (see [15]) that the slow surface S (or S_α) of system (1), (2) is the surface described by the equation

$$g(x, y, \alpha, 0) = 0. \tag{3}$$

Let $y = \phi(x, \alpha)$ be an isolated solution of Eq. 3. We call the subset S^s_α (S^u_α) of S defined by

$$\frac{\partial g}{\partial y}(x, \phi(x, \alpha), \alpha, 0) < 0 \quad (>0)$$

the stable (unstable) subset of S_α.

The set of irregular points (critical points of the projection of the slow surface onto the base) defined by

$$\frac{\partial g}{\partial y}(x, \phi(x, \alpha), \alpha, 0) = 0$$

on S_α is called the breakdown surface. Its dimension is equal to $n-1$. At all points of this surface the linearization of the fast subsystem (2) in a fiber has a zero eigenvalue [16].

In an ε-neighborhood of S^s_α (S^u_α) there exists a stable (unstable) slow invariant manifold $S^s_{\alpha,\varepsilon}$ ($S^u_{\alpha,\varepsilon}$). This means that the slow surface is an approximation of a slow invariant manifold (for $\varepsilon = 0$) [17].

The availability of the additional scalar parameter α provides the possibility of gluing the stable and unstable invariant manifolds at one point of the breakdown surface. The canard trajectory passes through this point. To explain this situation, consider the following system

$$\dot{x} = 1, \quad \dot{y} = 0, \quad \varepsilon\dot{z} = 2xz + \alpha - y,$$

where x, y and z are scalar variables, ε is a small positive parameter, α is a scalar parameter. The different canards are determined by

$$\dot{x} = 1, \ \ y = y_0, \ \ z = 0,$$

that is, each of them passes through the unique gluing point $x = 0, y = y_0, z = 0$ on the breakdown curve $x = 0$ of the slow surface $2xz + y_0 - y = 0$ for $\alpha = y_0$.

It should be noted that in the early papers devoted to canards in the case $\dim x = 1$, the existence of a unique canard corresponding to a unique value of the parameter $\alpha = \alpha^*$ was stated (more precisely, the "canard" value of parameter α^* exists on an interval of order $O(\mathrm{e}^{-1/\varepsilon})$). But in the case $\dim x > 1$ another picture is beginning to emerge. It was shown that a one-parameter family of canards exists [13].

2.2 Model with lumped parameters

Thermal explosion occurs when chemical reactions produce heat too rapidly for a stable balance between heat production and heat loss. The exothermic oxidation reaction is usually modeled as a single-step reaction obeying an Arrhenius temperature dependence. The first model for the self-ignition was constructed by Semenov in 1928 (see, for example [18]). The basic idea of the model is a competition between heat production in the reactant vessel (due to an exothermic reaction) and heat losses on the vessel's surface. Heat losses were assumed proportional to the temperature excess over the ambient temperature (Newtonian cooling). The main assumption was that there is no reactant conversion during the fast highly exothermic reaction. This assumption implies the absence of the energy-conservation law in the model. This gave the possibility of constructing an extremely simple and attractive mathematical model. Under spatial uniformity of the temperature we obtain the classical model of thermal explosion with reactant consumption in dimensionless form [2, 7]:

$$\varepsilon\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \eta(1-\eta)\exp\left(\theta/(1+\beta\theta)\right) - \alpha\theta, \tag{4}$$

$$\frac{\mathrm{d}\eta}{\mathrm{d}\tau} = \eta(1-\eta)\exp\left(\theta/(1+\beta\theta)\right), \tag{5}$$

$$\eta(0) = \eta_0/\left(1+\eta_0\right) = \bar{\eta}_0, \ \ \theta(0) = 0.$$

Here τ is the dimensionless time; θ and η are the dimensionless temperatures and the depth of conversion; η_0 is the criterion of autocatalyticity; the small parameters β and ε characterize the physical properties of the gas mixture and α is the dimensionless heat-loss parameter.

It should be noted that the system (4), (5) is singularly perturbed. According to the standard approach to such systems, the limiting case $\varepsilon \to 0$ is examined, and discontinuous solutions of the reduced system are analyzed. This makes it possible to determine some critical values of the initial conditions that provide a jump transition from the slow regime to the explosive one. The study of transitional regimes requires the application of higher approximations in the asymptotic analysis of the systems of the type given in Eqs. 4 and 5.

2.3 Model with distributed parameters

Consider nonlinear singularly perturbed parabolic system

$$\varepsilon \frac{\partial \theta}{\partial \tau} = \eta(1-\eta)\exp(\theta/(1+\beta\theta)) + \frac{1}{\delta} D_\xi \theta, \tag{6}$$

$$\varepsilon \frac{\partial \eta}{\partial \tau} = \varepsilon\,\eta(1-\eta)\exp(\theta/(1+\beta\theta)) + \frac{1}{\varrho} D_\xi \eta, \tag{7}$$

where

$$D_\xi(\cdot) = \frac{\partial^2(\cdot)}{\partial \xi^2} + \frac{n}{\xi}\frac{\partial(\cdot)}{\partial \xi}, \quad n = 0,1,2,$$

with boundary conditions

$$\left.\frac{\partial \theta}{\partial \xi}\right|_{\xi=0} = 0, \quad \left.\theta\right|_{\xi=1} = 0, \quad \left.\frac{\partial \eta}{\partial \xi}\right|_{\xi=0} = 0, \quad \left.\frac{\partial \eta}{\partial \xi}\right|_{\xi=1} = 0 \tag{8}$$

and initial conditions

$$\left.\theta\right|_{\tau=0} = 0, \quad \left.\eta\right|_{\tau=0} = \eta_0/(1+\eta_0).$$

This is a mathematical model for the problem of thermal explosion involving heat transfer and diffusion. Here θ is the dimensionless temperature; η is the dimensionless depth of conversion; τ is the dimensionless time; ε and β are small positive parameters; ϱ^{-1} is a constant diffusion coefficient; δ is a Frank–Kamenetsky criterion, that is, the scalar parameter characterizing the initial state of the system. Depending on the value of δ, reaction is either explosive or proceeds slowly. The value of the parameter δ separating slow and explosive regimes is called critical.

The critical value of δ is calculated as an asymptotic series in powers of the small parameter ε and the corresponding critical regimes are modeled by canards. For $n = 0$ (plane-parallel reactor), $n = 1$ (cylindrical reactor), $n = 2$ (spherical reactor), the corresponding values of δ are calculated.

2.4 Two-phase model of combustion

We now consider combustion models for a rarefied gas mixture in an inert porous or in a dusty medium. We assume that the temperature distribution and phase-to-phase heat exchange are uniform. The chemical-conversion kinetics are represented by a one-stage, irreversible reaction. The dimensionless model in the case of an autocatalytic reaction has the form [19]

$$\gamma\dot{\theta} = \eta(1-\eta)\exp(\theta/(1+\beta\theta)) - \alpha(\theta-\theta_c) - \kappa\theta, \tag{9}$$

$$\gamma_c\dot{\theta}_c = \alpha(\theta-\theta_c), \tag{10}$$

$$\dot{\eta} = \eta(1-\eta)\exp(\theta/(1+\beta\theta)), \tag{11}$$

$$\eta(0) = \eta_0/(1+\eta_0), \;\; \theta(0) = \theta_c(0) = 0.$$

Here, θ and θ_c are the dimensionless temperatures of the reactant and inert phases, respectively; η is the depth of conversion; η_0 is the criterion of autocatalyticity; the small parameters β and γ characterize the physical properties of gas mixture. The terms $-\kappa\theta$ and $-\alpha(\theta-\theta_c)$ reflect the external heat dissipation and phase-to-phase heat exchange. The parameter γ_c characterizes the physical features of the inert phase.

3 Classic model with lumped parameters

The system showing autocatalytic reaction features is

$$\varepsilon \frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \eta(1-\eta)\exp\left(\theta/(1+\beta\theta)\right) - \alpha\theta, \tag{12}$$

$$\frac{\mathrm{d}\eta}{\mathrm{d}\tau} = \eta(1-\eta)\exp\left(\theta/(1+\beta\theta)\right). \tag{13}$$

To simplify the demonstration of the main qualitative effects we employ a widely used assumption, $\beta = 0$, in thermal-explosion theory (more detailed analysis shows that the differences between the results obtained for cases $\beta = 0$ and $\beta \neq 0$ are not essential; see Sect. 5). For $\beta = 0$ the slow curve S_α of system (12), (13) is described by the equation

$$\eta(1-\eta)\mathrm{e}^{\theta} - \alpha\theta = 0.$$

The curve S_α has a different form depending on whether $\alpha > \mathrm{e}/4$ or $\alpha < \mathrm{e}/4$ (see Fig. 1). In the region $\theta < 1$ connected components of the curve S_α will be stable and in the region $\theta > 1$ they will be unstable. We shall denote a stable part S_α as S_α^s and an unstable part as S_α^u. There exist invariant manifolds $S_{\alpha,\varepsilon}^s$ and $S_{\alpha,\varepsilon}^u$ at a distance of $O(\varepsilon)$ from the curve S_α, corresponding to S_α^s and S_α^u.

We shall give a qualitative description of the behavior of the system (12), (13) for the changing parameter α. When $\alpha > \mathrm{e}/4$ the trajectories of the system in the phase plane move along the stable branch S_α^s and the value of θ does not exceed 1. These trajectories correspond to the slow regimes.

For $\alpha < \mathrm{e}/4$, the stable part S_α^s of the curve S_α consists of two separated branches and the system's trajectories, having reached the jump point at the tempo of the slow variable along S_α^s, jumps into the explosive regime.

Due to the continuous dependence of the right-hand sides of (12), (13) on the parameter α, we may assume that there are some intermediate trajectories in the region between those shown above in the neighborhood of $\alpha = \mathrm{e}/4$, and a critical one as well. With $\alpha = \mathrm{e}/4$ the slow curve S_α^s has a self-intersection point $(1, 1/2)$, and in this case it is possible to find the critical value of the parameter α in the form

$$\alpha = \alpha(\varepsilon) = \alpha_0 + \varepsilon\alpha_1 + \varepsilon^2\alpha_2 + \cdots, \quad \alpha_0 = \mathrm{e}/4. \tag{14}$$

There are two values of the parameter, namely $\alpha = \alpha^*$ and $\alpha = \alpha^{**}$, for which the trajectory of (12), (13) passes along the stable and the unstable parts of the slow curve for times that are not infinitesimally small.

The value $\alpha = \alpha^*$ corresponds to the canard, passing along the lower part of S_α^s and then along the upper part of S_α^u. The canard is taken as a mathematical object to model the critical trajectory, which corresponds

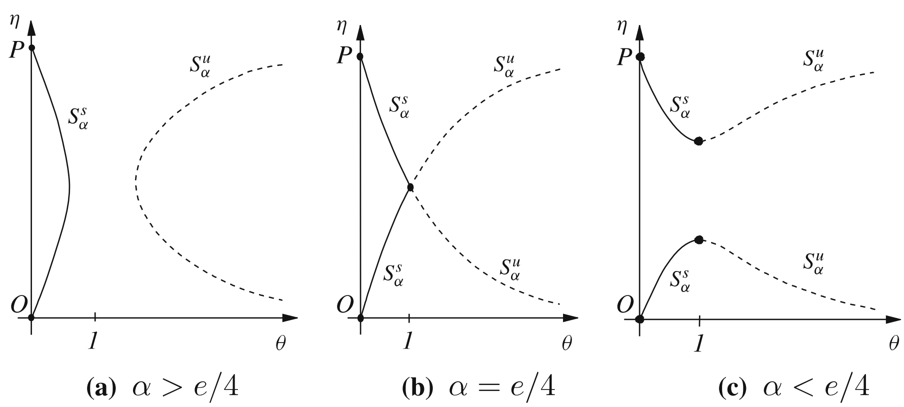
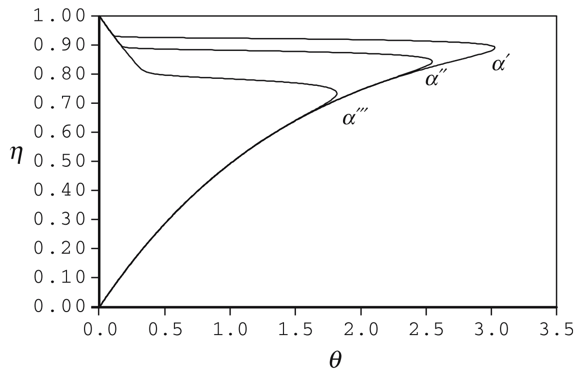


Fig. 1 The slow curve of the system (12), (13) in the case $\beta = 0$

Fig. 2 Canard trajectories of system for $\varepsilon = 0{\cdot}05$, $\alpha' = 0{\cdot}659941603$, $\alpha'' = 0{\cdot}659941646$, $\alpha''' = 0{\cdot}659952218$



to a chemical reaction separating the domain of self-acceleration reactions ($\alpha < \alpha^*$) and the domain of nonexplosive reactions ($\alpha > \alpha^*$).

The value $\alpha = \alpha^{**}$ is also important in a qualitative analysis of the system (12), (13). With $\alpha = \alpha^{**}$ there exist the trajectory (so-called false canard), passing along the lower part of S^u_α and then along the upper part of S^s_α. For $\alpha > \alpha^{**}$ we get a region of slow regimes and the trajectories of system (12), (13) will pass along the stable part of the slow curve.

Figure 2 shows numerical results for the canard trajectories of the system (12), (13) for α from the interval (α^*, α^{**}) $(\alpha^* < \alpha' < \alpha'' < \alpha''' < \alpha^{**})$.

The coefficients of the asymptotic series for α^* and α^{**} can be found by the methods of [10]. To calculate the critical value of the parameter $\alpha = \alpha^*$ we substitute (14) and the expression for corresponding canard

$$\eta = H(\theta, \varepsilon) \equiv H_0(\theta) + \varepsilon H_1(\theta) + \cdots$$

in (12), (13) and obtain

$$\left(H(\theta,\varepsilon)\left(1 - H(\theta,\varepsilon)\right) \mathrm{e}^{\theta} - \alpha(\varepsilon)\theta\right) H'(\theta,\varepsilon) = \varepsilon H(\theta,\varepsilon)\left(1 - H(\theta,\varepsilon)\right) \mathrm{e}^{\theta}$$

or, in more detailed form,

$$\left((H_0(\theta) + \varepsilon H_1(\theta) + \cdots)\,(1 - H_0(\theta) - \varepsilon H_1(\theta) - \cdots)\,\mathrm{e}^{\theta} - (\alpha_0 + \varepsilon\alpha_1 + \cdots)\theta\right)\left[H_0'(\theta) + \varepsilon H_1'(\theta) + \cdots\right]$$
$$= \varepsilon\,(H_0(\theta) + \varepsilon H_1(\theta) + \cdots)\,(1 - H_0(\theta) - \varepsilon H_1(\theta) - \cdots)\,\mathrm{e}^{\theta}.$$

Equating the coefficients of like powers of ε in the left and right sides of the last equation, we obtain

$$H_0(\theta) = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \alpha_0\theta\mathrm{e}^{-\theta}},$$

$$H_1(\theta) = \frac{\theta(\alpha_1 H_0' + \alpha_0)}{H_0'(1 - 2H_0)\mathrm{e}^{\theta}},$$

$$H_2(\theta) = \frac{\theta\left(\alpha_1 H_1' + \alpha_2 H_0'\right) + H_0' H_1^2 \mathrm{e}^{\theta} + H_1\left(1 - H_1'\right)(1 - 2H_0)\,\mathrm{e}^{\theta}}{H_0'(1 - 2H_0)\mathrm{e}^{\theta}}.$$

The coefficients in the expression (14) α_i $(i = 0, 1, 2, \ldots)$ are found from the continuity of the functions $H_i = H_i(\theta)$ at $\theta = 1$. Thus, we have

$$\alpha^* = \mathrm{e}/4(1 - 2\sqrt{2}\varepsilon - 49/9\varepsilon^2) + O(\varepsilon^3).$$

For $\beta \neq 0$ we obtain the following approximate formula

$$\alpha^* = (1 - \beta)\mathrm{e}/4(1 - 2\sqrt{2}\varepsilon) + \cdots.$$

The value $\alpha = \alpha^{**}$ and corresponding false canard can be found in the same way. For this case we obtain

$\alpha^{**} = \mathrm{e}/4(1 + 2\sqrt{2}\varepsilon - 49/9\varepsilon^2) + O(\varepsilon^3)$.

4 Model with distributed parameters

The approach suggested may also be applied for the calculation of the critical value of the parameter in the case of a model with distributed parameters, taking into consideration the processes of thermal conductivity and diffusion. A one-dimensional slow invariant manifold corresponds to the critical regime. This manifold can be found in parametric form as follows:

$$\begin{aligned} \theta &= \Theta(v,\xi,\varepsilon) = \theta_0(v,\xi) + \varepsilon\theta_1(v,\xi) + O(\varepsilon^2), \\ \eta &= H(v,\xi,\varepsilon) = \eta_0(v,\xi) + \varepsilon\eta_1(v,\xi) + O(\varepsilon^2), \\ \frac{\mathrm{d}v}{\mathrm{d}\tau} &= V(v,\varepsilon) = V_0(v) + \varepsilon V_1(v) + O(\varepsilon^2). \end{aligned} \tag{15}$$

The coefficient δ will also be found as an asymptotical expansion:

$$\delta = \delta(\varepsilon) = \delta_0(1 + \varepsilon\delta_1) + O(\varepsilon^2). \tag{16}$$

The functions Θ and H satisfy the equations

$$\varepsilon\frac{\partial\Theta}{\partial v}V = H(1-H)\exp\left(\Theta/(1+\beta\Theta)\right) + \frac{1}{\delta(\varepsilon)}D_\xi\Theta, \tag{17}$$

$$\varepsilon\frac{\partial H}{\partial v}V = \varepsilon H(1-H)\exp\left(\Theta/(1+\beta\Theta)\right) + \frac{1}{\varrho}D_\xi H, \tag{18}$$

and the boundary conditions

$$\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=0} = 0, \quad \left.\Theta\right|_{\xi=1} = 0, \quad \left.\frac{\partial H}{\partial\xi}\right|_{\xi=0} = 0, \left.\frac{\partial H}{\partial\xi}\right|_{\xi=1} = 0, \tag{19}$$

which are obtained by substituting θ, η, δ by $\Theta, H, \delta(\varepsilon)$ in (6)–(8).

Substituting (15)–(16) in (17)–(19) and setting $\varepsilon = 0$, we obtain

$$0 = H_0(1-H_0)\exp\left(\Theta_0/(1+\beta\Theta_0)\right) + \frac{1}{\delta_0}D_\xi\Theta_0, \tag{20}$$

$$\left.\frac{\partial\Theta_0}{\partial\xi}\right|_{\xi=0} = 0, \quad \left.\Theta_0\right|_{\xi=1} = 0, \tag{21}$$

$$0 = \frac{1}{\varrho}D_\xi H_0, \tag{22}$$

$$\left.\frac{\partial H_0}{\partial\xi}\right|_{\xi=0} = 0, \quad \left.\frac{\partial H_0}{\partial\xi}\right|_{\xi=1} = 0 \tag{23}$$

Equations (20), (22) with boundary conditions (21), (23) make it possible to determine Θ_0 and $H_0 = v$. The condition of self-intersection of Θ_0 at $v = 1/2$ allows us to get δ_0.

The functions Θ_1 and H_1 are defined by the equations

$$\begin{aligned} V_0\frac{\partial\Theta_0}{\partial v} &= H_0(1-H_0)\exp\left(\Theta_0/(1+\beta\Theta_0)\right)\Theta_1/(1+\beta\Theta_0)^2 \\ &\quad + (1-2H_0)\exp(\Theta_0/(1+\beta\Theta_0))H_1 - \frac{\delta_1}{\delta_0}D_\xi\Theta_0 + \frac{1}{\delta_0}D_\xi\Theta_1, \end{aligned} \tag{24}$$

$$V_0 = H_0(1 - H_0)\exp(\Theta_0/(1+\beta\Theta_0)) + \frac{1}{\varrho} D_\xi H_1, \tag{25}$$

and the boundary conditions

$$\left.\frac{\partial\Theta_1}{\partial\xi}\right|_{\xi=0} = 0, \quad \left.\Theta_1\right|_{\xi=1} = 0, \tag{26}$$

$$\left.\frac{\partial H_1}{\partial\xi}\right|_{\xi=0} = 0, \quad \left.\frac{\partial H_1}{\partial\xi}\right|_{\xi=1} = 0. \tag{27}$$

Integrating by parts (25) and taking into consideration (27), we obtain

$$V_0 \;=\; (n+1)v(1-v)\int_0^1 \exp(\Theta_0/(1+\beta\Theta_0))\mathrm{d}\xi.$$

To find δ_1 we consider the linear boundary-value problem (24), (26) for $v = 1/2$. Let $\psi(\xi)$ be the eigenfunction of the corresponding homogeneous boundary-value problem. The existence condition of the nonhomogeneous problem (24), (26) for $v = 1/2$ has the form

$$\int_0^1 \left.\left[\frac{\partial\Theta_0}{\partial v} V_0 - (1-2v)\exp(\Theta_0/(1+\beta\Theta_0))H_1 + \frac{\delta_1}{\delta_0} D_\xi\Theta_0\right]\right|_{v=1/2} \mathrm{d}\xi.$$

From the last equation we obtain

$$\delta_1 = -\int_0^1 \left.\frac{\partial\Theta_0}{\partial v} V_0\right|_{v=1/2} \mathrm{d}\xi \Big/ \int_0^1 \left. D_\xi\Theta_0\right|_{v=1/2} \mathrm{d}\xi.$$

Note that function H_1 is superfluous in our consideration. This implies that the value δ_1 is independent of ϱ, like δ_0.

Thus

$$\delta^* = \delta_0(1 + |\delta_1|\varepsilon + O(\varepsilon^2))$$

corresponds to a canard and gives the required critical condition for a thermal explosion. The value

$$\delta^{**} = \delta_0(1 - |\delta_1|\varepsilon + O(\varepsilon^2))$$

corresponds to a false canard. The interval (δ^*, δ^{**}) corresponds to transitional combustion regimes. For the difference of the values δ^* and δ^{**} we have

$$\delta^* - \delta^{**} = 2\delta_0|\delta_1|\varepsilon + O(\varepsilon^2).$$

Now we put $\beta = 0$ and give the results of our calculations for plane-parallel ($n = 0$), cylindrical ($n = 1$) and spherical ($n = 2$) reactors:

$$\begin{array}{llll} n = 0, & \delta_0 = 3{\cdot}5138, & |\delta_1| = 2{\cdot}22, & \delta^* - \delta^{**} \simeq 15{\cdot}58\varepsilon, \\ n = 1, & \delta_0 = 8, & |\delta_1| = 16\frac{4-\pi}{4+\pi} \simeq 1{\cdot}92, & \delta^* - \delta^{**} \simeq 30{\cdot}77\varepsilon, \\ n = 2, & \delta_0 = 13{\cdot}32, & |\delta_1| = 1{\cdot}74, & \delta^* - \delta^{**} \simeq 46{\cdot}35\varepsilon. \end{array}$$

5 Gas combustion in an inert medium

In the absence of external heat dissipation ($\kappa = 0$) the system of differential Eqs. (9)–(11) possesses a first integral, namely

$$\eta - \gamma\theta - \gamma_c\theta_c = \bar{\eta}_0,$$

and therefore we obtain

$$\gamma \frac{d\theta}{d\tau} = \eta(1-\eta)\exp\left(\frac{\theta}{1+\beta\theta}\right) - \alpha\left(1+\frac{\gamma}{\gamma_c}\right)\theta + \frac{\alpha}{\gamma_c}(\eta - \bar{\eta}_0), \tag{28}$$

$$\frac{d\eta}{d\tau} = \eta(1-\eta)\exp\left(\frac{\theta}{1+\beta\theta}\right) \tag{29}$$

with initial conditions

$$\eta(0) = \bar{\eta}_0, \quad \theta(0) = 0.$$

The dependence of the slow curve S_α

$$F(\eta,\theta,\alpha) = \eta(1-\eta)\exp\left(\frac{\theta}{1+\beta\theta}\right) - \alpha\left(\theta - \frac{\eta-\bar{\eta}_0}{\gamma_c}\right) = 0$$

on the relation between parameter values gives different forms (see Fig. 3).

We take α as control parameter with fixed γ_c. The point $\theta = \theta^*$, $\eta = \eta^*$ is the self-intersection point of the slow curve at $\alpha = \alpha_0$. Here, $\alpha = \alpha_0$, $\theta = \theta^*$, $\eta = \eta^*$ satisfy the system

$$F(\eta,\theta,\alpha) = F_\eta(\eta,\theta,\alpha) = F_\theta(\eta,\theta,\alpha) = 0.$$

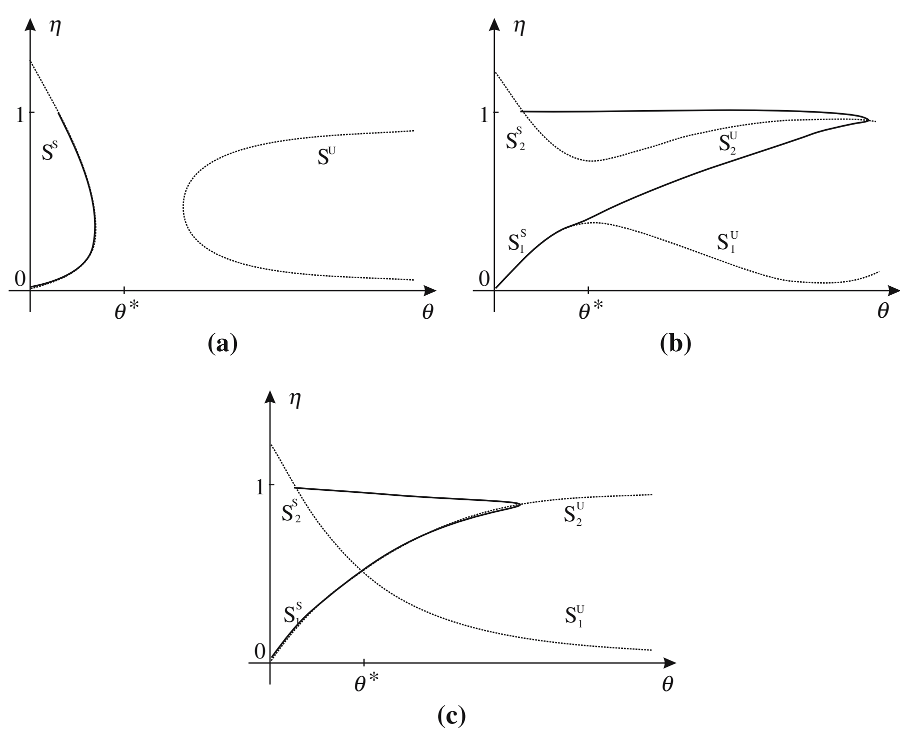


Fig. 3 The trajectories (the solid line) of the system (28), (29) and the slow curve (the dashed line) in various cases

For $\alpha > \alpha_0$ each set S^s_α and S^u_α of S_α consists of a single connected curve; see Fig. 3(a). Hence the system has an stable invariant manifold $S^s_{\alpha,\gamma}$ and an unstable invariant manifold $S^u_{\alpha,\gamma}$ near S^s_α and S^u_α, respectively.

Since the initial point $(0, \bar{\eta}_0)$ belongs to the basin of attraction of the set $S^s_{\alpha,\gamma}$, after a short time the trajectory follows the stable slow invariant manifold $S^s_{\alpha,\gamma}$ and tends to the equilibrium $P((1-\bar{\eta}_0)/(\gamma+\gamma_c), 1)$ as t tends to ∞. This behavior corresponds to the slow combustion regime; see Fig. 4.

For $\alpha < \alpha_0$ each set S^s_α and S^u_α consists of two different components (Fig. 3(b)) and the system has an stable invariant manifold $S^s_{\alpha,\gamma}$ (unstable invariant manifold $S^u_{\alpha,\gamma}$) near each component of S^s_α (S^u_α). For γ sufficiently small and after a short time, the solution will follow the component of $S^s_{\alpha,\gamma}$ to breakdown point. After this time, $\theta(t)$ will increase rapidly. This behavior characterizes the explosive regime; see Fig. 5.

The transition region from the slow to the explosive regime exists due to the continuous dependence of our system on the parameters α and γ_c ($\gamma_c > 0$). In this special case ($\alpha = \alpha_0$) the slow curve has an intersection point (θ^*, η^*); see Fig. 3(c). Here the system has a stable invariant manifold $S^s_{\alpha,\gamma}$ (unstable invariant manifold $S^u_{\alpha,\gamma}$) near each component of the slow curve S^s_α (S^u_α).

We can observe the existence of canard solutions that describe the following regime: the temperature increases as high as is possible but without explosion (see Fig. 6); this may be the aim of technological process. We note that this regime is critical, and corresponds to a chemical reaction separating the domain of self-accelerating reactions and the domain of slow reactions.

We can find the canard solution and corresponding value of α by the following asymptotic expansions

$$\alpha^* = \alpha(\gamma) = \alpha_0 + \gamma\alpha_1 + \gamma^2\alpha_2 + \cdots,$$
$$\eta = H(\theta, \gamma) = H_0(\theta) + \gamma H_1(\theta) + \gamma^2 H_2(\theta) + \cdots.$$

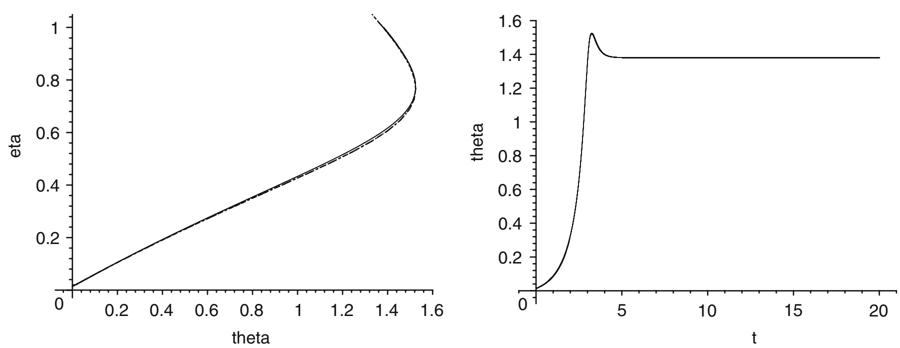


Fig. 4 The trajectory of the system (28), (29) and the temperature–time characteristics for $\alpha = 1{\cdot}4$, $\beta = 0{\cdot}1$, $\gamma = 0{\cdot}01$, $\gamma_c = 0{\cdot}7$, $\bar{\eta}_0 = 0{\cdot}02$

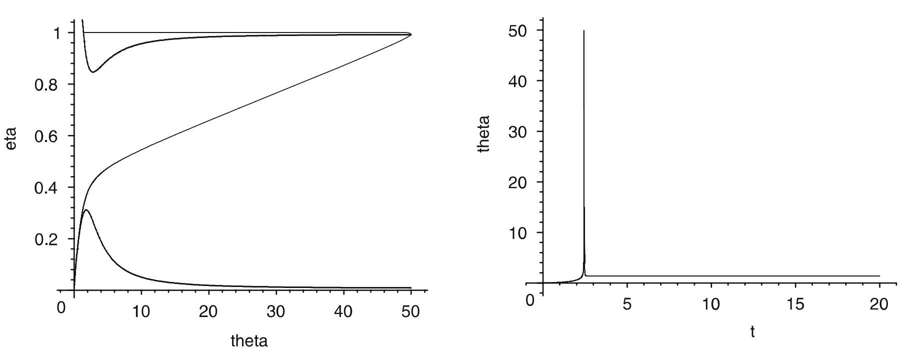


Fig. 5 The trajectory of the system (28), (29) and the temperature–time characteristics for $\alpha = 0{\cdot}7$, $\beta = 0{\cdot}1$, $\gamma = 0{\cdot}01$, $\gamma_c = 1{\cdot}1$, $\bar{\eta}_0 = 0{\cdot}02$

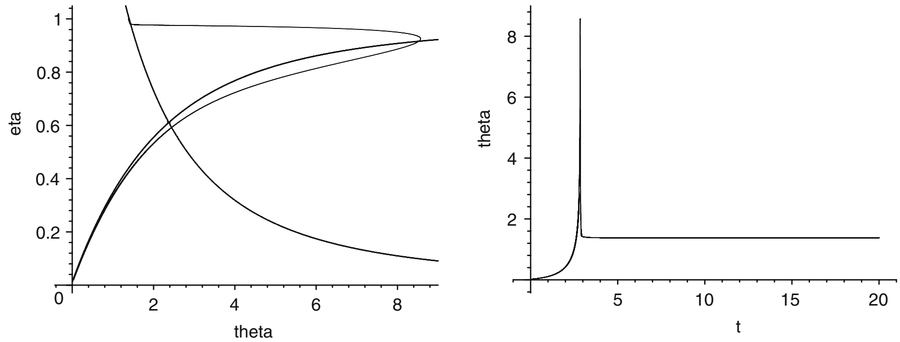


Fig. 6 The trajectory of the system (28), (29) and the temperature–time characteristics for $\alpha = \alpha^* = 0{\cdot}949$, $\beta = 0{\cdot}1$, $\gamma = 0{\cdot}01$, $\gamma_c = 0{\cdot}7$, $\bar{\eta}_0 = 0{\cdot}02$

We substitute these expansions in (28), (29) and obtain

$$\left(H(\theta,\gamma)\,(1-H(\theta,\gamma))\exp\left(\frac{\theta}{1+\beta\theta}\right)-\alpha(\gamma)\left(1+\frac{\gamma}{\gamma_c}\right)\theta+\frac{\alpha(\gamma)}{\gamma_c}(H(\theta,\gamma)-\bar{\eta}_0)\right)H'(\theta,\gamma)$$
$$=\gamma H(\theta,\gamma)\,(1-H(\theta,\gamma))\exp\left(\frac{\theta}{1+\beta\theta}\right)$$

or, in more detailed form,

$$\Bigg((H_0(\theta)+\gamma H_1(\theta)+\cdots)\,(1-H_0(\theta)-\gamma H_1(\theta)-\cdots)\exp\left(\frac{\theta}{1+\beta\theta}\right)-(\alpha_0+\gamma\alpha_1+\cdots)\left(1+\frac{\gamma}{\gamma_c}\right)\theta$$
$$+\frac{(\alpha_0+\gamma\alpha_1+\cdots)}{\gamma_c}(H_0(\theta)+\gamma H_1(\theta)+\cdots-\bar{\eta}_0)\Bigg)\left[H_0'(\theta)+\gamma H_1'(\theta)+\cdots\right]$$
$$=\gamma\,(H_0(\theta)+\gamma H_1(\theta)+\cdots)\,(1-H_0(\theta)-\gamma H_1(\theta)-\cdots)\exp\left(\frac{\theta}{1+\beta\theta}\right).$$

Equating the coefficients of like powers of γ in the left and right parts of the last equation and using the continuity condition for the functions $H_i = H_i(\theta)$ $(i = 0, 1, 2, \ldots)$ at $\theta = \theta^*$, we obtain

$$\alpha_0 = \gamma_c(2\eta^*-1)\exp\left(\frac{\theta^*}{1+\beta\theta^*}\right),$$
$$\alpha_1 = -\alpha_0\left[\frac{\theta^*}{\gamma_c\theta^*-\eta^*+\bar{\eta}_0}+\frac{-1+2\eta^*+\sqrt{(1-2\eta^*)^2+2\eta^*(1-\eta^*)\,(1-2\beta(1+\beta\theta^*))}}{\eta^*(1-\eta^*)\,(1-2\beta(1+\beta\theta^*))}(1+\beta\theta^*)^2\right].$$

Here, $\theta = \theta^*$ is a root of the equation

$$\gamma_c\,(1+\beta\theta)^4 = \gamma_c\theta^2-\theta(1-2\bar{\eta}_0)+\gamma_c^{-1}(\bar{\eta}_0^2-\bar{\eta}_0),$$

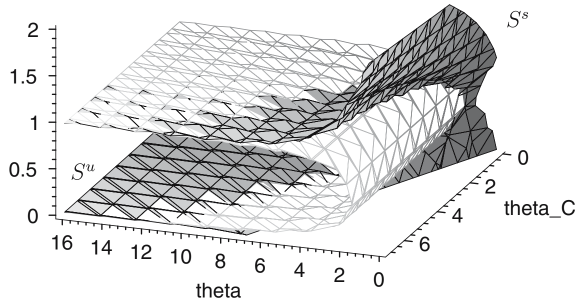
and $\eta^* = H_0(\theta^*)$, where the function $H_0 = H_0(\theta)$ is determined by

$$H_0(1-H_0)\exp\left(\frac{\theta}{1+\beta\theta}\right)-\alpha_0\theta+\alpha_0\frac{H_0-\bar{\eta}_0}{\gamma_c} = 0.$$

For $\beta = 0$ we have

$$\theta^*|_{\beta=0} = \theta_0^* = \frac{1}{2}\left(\gamma_c^{-1}(1-2\bar{\eta}_0)+\sqrt{4+\gamma_c^{-2}}\right),$$
$$\eta^*|_{\beta=0} = \frac{1}{2}\left(1+\sqrt{1+4\gamma_c^2}\right)-\gamma_c, \qquad \alpha_0|_{\beta=0} = \frac{\exp\theta_0^*}{2+\sqrt{4+\gamma_c^{-2}}}.$$

Fig. 7 The slow surface (the dark area) and the surface of irregular points (the light area) of the system (9)–(11)



For example, in the case $\bar{\eta}_0 = 0$, the asymptotic expansion of the canard value of parameter α is [13, 14] (we take the zero-approximation term with order $O(\beta)$ and the first-approximation term with order $O(\gamma)$):

$$\alpha^* = \frac{1-\beta\theta_0^{*2}}{2+\sqrt{4+\gamma_c^{-2}}}\mathrm{e}^{\theta_0^*}\left[1-\gamma\left(\frac{1}{2}\gamma_c^{-2}+\frac{1}{2}\gamma_c^{-1}(2+\sqrt{4+\gamma_c^{-2}})+\sqrt[4]{4+\gamma_c^{-2}}\sqrt{2+\sqrt{4+\gamma_c^{-2}}}\right)\right],$$

$$\theta_0^* = \frac{1}{2}\left(\gamma_c^{-1}+\sqrt{4+\gamma_c^{-2}}\right).$$

For $\kappa \neq 0$ we obtain the problem for constructing the critical trajectory in R^3. The breakdown curve separates the stable subset (S^s) of the slow surface S and the unstable one (S^u); see Fig. 7. Here S is described by the equation

$$F(\eta,\theta,\theta_c,\alpha) = \eta(1-\eta)\exp\left(\frac{\theta}{1+\beta\theta}\right)-\alpha(\theta-\theta_c)-\kappa\theta = 0,$$
$$S^s = \{(\eta,\theta,\theta_c):\ F_\theta(\eta,\theta,\theta_c,\alpha) < 0\},$$
$$S^u = \{(\eta,\theta,\theta_c):\ F_\theta(\eta,\theta,\theta_c,\alpha) > 0\}.$$

The different types of chemical regimes take place depending on the relation between values of the parameters; see Figs. 8 and 9. We shall use a canard as *a separating solution* corresponding to the critical regime of the chemical reaction. We can find the canard solution and the corresponding value of α by the following asymptotic expansions

$$\alpha^* = \alpha(\gamma) = \alpha_0 + \gamma\alpha_1 + \gamma^2 \dots ,$$
$$\theta = \theta(\eta,\gamma) = \phi_0(\eta) + \gamma\phi_1(\eta) + \gamma^2 \dots ,$$
$$\theta_c = \theta_c(\eta,\gamma) = \psi_0(\eta) + \gamma\psi_1(\eta) + \gamma^2 \dots .$$

From the system (9)–(11) we have

$$\gamma\theta'(\eta,\gamma)\eta\,(1-\eta)\exp\left(\frac{\theta(\eta,\gamma)}{1+\beta\theta(\eta,\gamma)}\right)$$
$$= \eta\,(1-\eta)\exp\left(\frac{\theta(\eta,\gamma)}{1+\beta\theta(\eta,\gamma)}\right) - \alpha(\gamma)\,(\theta(\eta,\gamma)-\theta_c(\eta,\gamma)) - \kappa\theta(\eta,\gamma),$$
$$\gamma_c\theta_c'(\eta,\gamma)\eta\,(1-\eta)\exp\left(\frac{\theta(\eta,\gamma)}{1+\beta\theta(\eta,\gamma)}\right) = \alpha(\gamma)\,(\theta(\eta,\gamma)-\theta_c(\eta,\gamma)).$$

Substituting the asymptotic expansions for $\alpha(\gamma)$, $\theta(\eta,\gamma)$, $\theta_c(\eta,\gamma)$ in the last relationships and equating the coefficients of like powers of γ in the left and right parts, we obtain the equations for the functions $\phi_i = \phi_i(\eta)$ and $\psi_i = \psi_i(\eta)$ $(i = 0,1,2,\dots)$. The coefficients α_i are found from the continuity condition for

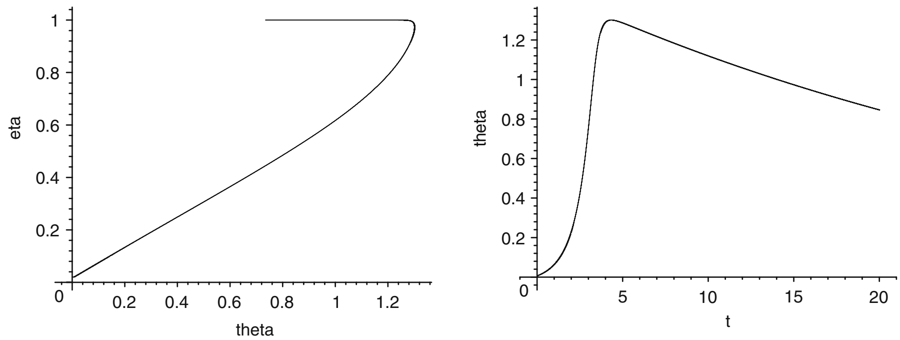


Fig. 8 The trajectory of the system (9)–(11) and the temperature–time characteristics in the case of slow regime: $\alpha = 3$, $\beta = 0{\cdot}1$, $\gamma = 0{\cdot}01$, $\gamma_c = 0{\cdot}7$, $\bar{\eta}_0 = 0{\cdot}02$, $\kappa = 0{\cdot}02$

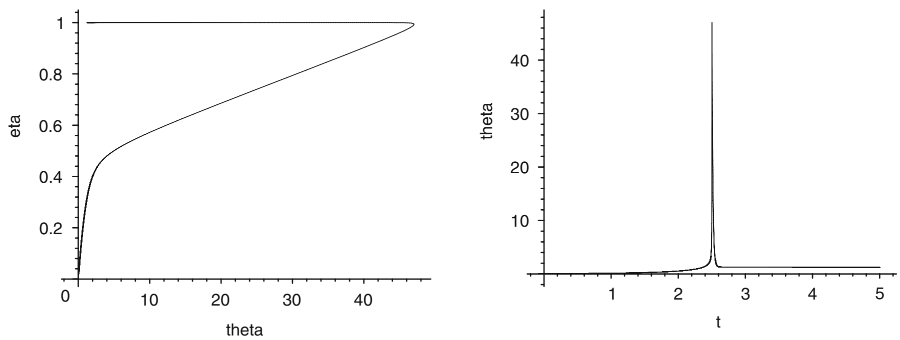


Fig. 9 The trajectory of the system (9)–(11) and the temperature–time characteristics in the case of thermal explosion: $\alpha = 0{\cdot}7$, $\beta = 0{\cdot}1$, $\gamma = 0{\cdot}01$, $\gamma_c = 0{\cdot}7$, $\bar{\eta}_0 = 0{\cdot}02$, $\kappa = 0{\cdot}02$

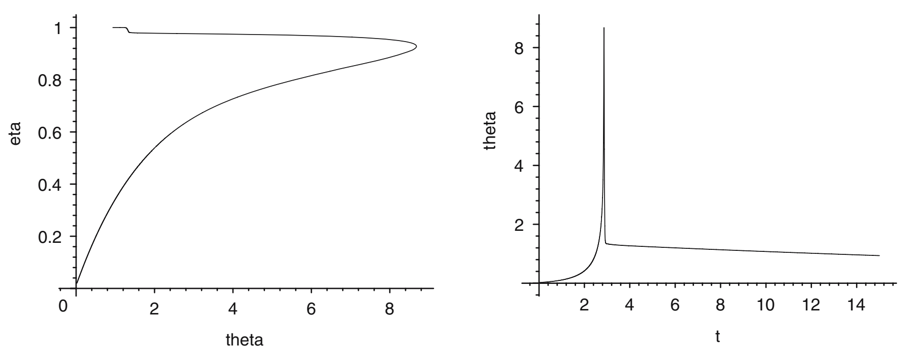


Fig. 10 The trajectory of the system (9)–(11) and the temperature–time characteristics in the case of critical regime for $\alpha = \alpha^* = 0{\cdot}9033$, $\beta = 0{\cdot}1$, $\gamma = 0{\cdot}01$, $\gamma_c = 0{\cdot}7$, $\bar{\eta}_0 = 0{\cdot}02$, $\kappa = 0{\cdot}02$

these functions at $\eta = \eta^*$. The equations

$$\eta(1-\eta)\exp\left(\frac{\phi_0}{1+\beta\phi_0}\right) - \alpha_0\left(\phi_0 - \psi_0\right) - \kappa\phi_0 = 0,$$

$$\gamma_c\psi_0'\eta(1-\eta)\exp\left(\frac{\phi_0}{1+\beta\phi_0}\right) = \alpha_0\left(\phi_0-\psi_0\right), \quad \psi_0(\bar{\eta}_0) = 0,$$

$$\eta^*(1-\eta^*)\exp\left(\frac{\phi_0(\eta^*)}{1+\beta\phi_0(\eta^*)}\right)\frac{1}{(1+\beta\phi_0(\eta^*))^2} - (\alpha_0+\kappa) = 0,$$

$$(1-2\eta^*)\exp\left(\frac{\phi_0(\eta^*)}{1+\beta\phi_0(\eta^*)}\right) + \alpha_0\psi_0'(\eta^*) = 0$$

define the value α_0 and the functions $\phi_0 = \phi_0(\eta)$ and $\psi_0 = \psi_0(\eta)$. For the determination of α_1 and the functions $\phi_1 = \phi_1(\eta)$ and $\psi_1 = \psi_1(\eta)$ we have

$$\phi_0'\eta(1-\eta)\exp\left(\frac{\phi_0}{1+\beta\phi_0}\right)$$
$$= \left[\eta(1-\eta)\exp\left(\frac{\phi_0}{1+\beta\phi_0}\right)\frac{1}{(1+\beta\phi_0)^2} - (\alpha_0+\kappa)\right]\phi_1 + \alpha_0\psi_1 - \alpha_1(\phi_0-\psi_0),$$
$$\eta(1-\eta)\exp\left(\frac{\phi_0}{1+\beta\phi_0}\right)\left[\phi_0' + \gamma_c\psi_1' + \frac{\phi_1(\gamma_c\psi_0'-1)}{(1+\beta\phi_0)^2}\right] = -\kappa\phi_1, \quad \psi_1(\bar{\eta}_0) = 0,$$
$$\alpha_1 = \frac{1}{\phi_0(\eta^*) - \psi_0(\eta^*)}\left[\alpha_0\psi_1(\eta^*) - \phi_0'(\eta^*)\eta^*(1-\eta^*)\exp\left(\frac{\phi_0(\eta^*)}{1+\beta\phi_0(\eta^*)}\right)\right].$$

The effect of external cooling may be observed: for $\kappa \neq 0$ the critical value of the parameter $\alpha = \alpha^*$ decreases; see Figs. 6 and 10.

This approach was used in [20] for a first-order reaction.

6 Conclusion

Singularly perturbed systems of differential equations describing a thermal explosion have been analyzed. Critical and transient regimes were modeled by means of a geometric theory of singular perturbations methods. The mathematical objects were introduced for first-order reactions and for an autocatalytic case. These objects make it possible to follow the continuous transition of a reaction from the slow to the explosive regime. The critical regime is modeled by a mathematical object called canard in the modern mathematical literature. Such trajectories pass from the stable slow invariant manifold to the unstable one. The system's trajectories, passing some part of its way along critical trajectories, belong to the transient regimes. Thus, the transient region is separated into a region of slow transient regimes and a region of explosive transient regimes. The asymptotic formulae for the calculation of the critical values of the heat-loss parameter were obtained. This approach was extended to combustion models with distributed parameters. It should be noted that such an approach was used in [21] to describe canard traveling waves.

Acknowledgements The authors are grateful to a number of colleagues for their encouragement and suggestions. Special thanks are given to Grigory Barenblatt, Vladislav Babkin, Valery Babushok and Vladimir Gol'dshtein. E. Shchepakina and V. Sobolev also wish to thank the Boole Centre for Research in Informatics and Department of Applied Mathematics, University College Cork, Ireland, for their hospitality and financial support.

Appendix. Elements of the geometric theory of singularly perturbed systems

A.1 Slow integral manifolds

Consider the ordinary differential system

$$\frac{dx}{dt} = f(x,y,t,\varepsilon),$$
$$\varepsilon\frac{dy}{dt} = g(x,y,t,\varepsilon), \tag{30}$$

with vector variables x and y, and a small positive parameter ε. The usual approach in the qualitative study of (30) is to consider first the degenerate system

$$\frac{dx}{dt} = f(x,y,t,0),$$
$$0 = g(x,y,t,0),$$

and then to draw conclusions about the qualitative behavior of the full system (30) for sufficiently small ε. A special case of this approach is the quasi-steady-state assumption. A mathematical justification of that method can be given by means of the theory of integral manifolds for singularly perturbed systems (30). In order to recall a basic result of the geometric theory of singularly perturbed systems, we introduce the following terminology and assumptions.

The system of equations

$$\frac{dx}{dt} = f(x, y, t, \varepsilon) \tag{31}$$

represents the slow subsystem, and the system of equations

$$\varepsilon \frac{dy}{dt} = g(x, y, t, \varepsilon) \tag{32}$$

the fast subsystem, so it is natural to call (31) the slow subsystem and (32) the fast subsystem of system (30). In the present paper we use a method for the qualitative asymptotic analysis of differential equations with singular perturbations. The method relies on the theory of integral manifolds, which essentially replaces the original system by another on an integral manifold with dimension equal to that of the slow subsystem. In the zero-epsilon approximation ($\varepsilon = 0$), this method leads to a modification of the quasi-steady-state approximation. Recall, that a smooth surface S in $R^m \times R^n \times R$ is called an integral manifold of the system (30) if any trajectory of the system having at least one point in common with S lies entirely in S. Formally, if $(x(t_0), y(t_0), t_0) \in S$, then the trajectory $(x(t, \varepsilon), y(t, \varepsilon), t)$ lies entirely in S. An integral manifold of an autonomous system

$$\dot{x} = f(x, y, \varepsilon),$$
$$\varepsilon \dot{y} = g(x, y, \varepsilon)$$

has the form $S_1 \times (-\infty, \infty)$, where S_1 is a surface in the phase space $R^m \times R^n$. The only integral manifolds of system (30) of relevance here are those of dimension m (the dimension of the slow variables) that can be represented as the graphs of vector- valued functions

$$y = h(x, t, \varepsilon).$$

We also stipulate that $h(x, t, 0) = h^{(0)}(x, t)$, where $h^{(0)}(x, t)$ is a function whose graph is a sheet of the slow surface, and we assume that $h(x, t, \varepsilon)$ is a sufficiently smooth function of ε. In autonomous systems the integral manifolds will be graphs of functions

$$y = h(x, \varepsilon).$$

Such integral manifolds are called manifolds of slow motions—the origin of this term lies in nonlinear mechanics. An integral manifold may be regarded as a surface on which the phase velocity has a local minimum, that is, a surface characterized by the most persistent phase changes (motions). Integral manifolds of slow motions constitute a refinement of the sheets of the slow surface, obtained by taking the small parameter ε into consideration.

The motion along an integral manifold is governed by the equation

$$\dot{x} = f(x, h(x, t, \varepsilon), t, \varepsilon).$$

If $x(t, \varepsilon)$ is a solution of this equation, then the pair $\big(x(t, \varepsilon), y(t, \varepsilon)\big)$, where $y(t, \varepsilon) = h(x(t, \varepsilon), t, \varepsilon)$, is a solution of the original system (30), since it defines a trajectory on the integral manifold.

Consider the *boundary-layer* subsystem, that is,

$$\frac{dy}{d\tau} = g(x, y, t, 0), \quad \tau = t/\varepsilon,$$

treating x and t as parameters. We shall assume that some of the steady states $y^0 = y^0(x, t)$ of this subsystem are asymptotically stable and that a trajectory starting at any point of the domain approaches one of these states as closely as desired as $t \to \infty$. This assumption will hold, for example, if the matrix

$$(\partial g/\partial y)(x, y^0(x,t), t, 0)$$

is stable for part of the stationary states and the domain can be represented as the union of the domains of attraction of the asymptotically stable steady states.

Let I_i be the interval $I_i := \{\varepsilon \in R\colon 0 < \varepsilon < \varepsilon_i\}$, where $0 < \varepsilon_i \ll 1, i = 0, 1, \ldots$.

(i) $f\colon R^m \times R^n \times R \times \overline{I_0} \to R^m$, $g\colon R^m \times R^n \times R \times \overline{I_0} \to R^n$ are sufficiently smooth and uniformly bounded together with their derivatives.

(ii) There is some region $G \in R^m$ and a map $h\colon G \times R \to R^m$ of the same smoothness as g such that

$$g(x, h(x,t), t, 0) \equiv 0, \quad \forall (x,t) \in G \times R.$$

(iii) The spectrum of the Jacobian matrix $g_y(x, h(x,t), t, 0)$ is uniformly separated from the imaginary axis for all $(x,t) \in G \times R$.

Then the following result is valid:

Proposition *Under the assumptions* (i)–(iii) *there is a sufficiently small positive* ε_1, $\varepsilon_1 \le \varepsilon_0$, *such that, for* $\varepsilon \in \overline{I_1}$, *system* (30) *has a smooth integral manifold* $\mathcal{M}_\varepsilon$ *with the representation*

$$\mathcal{M}_\varepsilon := \{(x, y, t) \in R^{n+m+1}\colon y = \psi(x, t, \varepsilon),\ (x,t) \in G \times R\}.$$

Remark The global boundedness assumption in (i) with respect to (x, y) can be relaxed by modifying f and g outside some bounded region of $R^n \times R^m$.

A.2 Asymptotic representation of integral manifolds

When the method of integral manifolds is being used to solve a specific problem, a central question is the calculation of the function $h(x, t, \varepsilon)$ in terms of the manifold described. An exact calculation is generally impossible, and various approximations are necessary. One possibility is the asymptotic expansion of $h(x, t, \varepsilon)$ in integer powers of the small parameter:

$$h(x,t,\varepsilon) = h_0(x,t) + \varepsilon h_1(x,t) + \cdots + \varepsilon^k h_k(x,t) + \cdots.$$

Substituting this formal expansion in Eq. (32) i.e.,

$$\varepsilon \frac{\partial h}{\partial t} + \varepsilon \frac{\partial h}{\partial x} f(x, h(x,t,\varepsilon), t, \varepsilon) = g(x, h, \varepsilon),$$

we obtain the relationship

$$\varepsilon \sum_{k\ge 0} \varepsilon^k \frac{\partial h_k}{\partial t} + \varepsilon \sum_{k \ge 0} \varepsilon^k \frac{\partial h_k}{\partial x} f\left(x, \sum_{k\ge 0} \varepsilon^k h_k, t, \varepsilon\right) = g\left(x, \sum_{k \ge 0} \varepsilon^k h_k, t, \varepsilon\right). \tag{33}$$

We use the formal asymptotic representations

$$f\left(x, \sum_{k\ge 0} \varepsilon^k h_k, t, \varepsilon\right) = \sum_{k \ge 0} \varepsilon^k f^{(k)}(x, h_0, \ldots, h_{k-1}, t),$$

and

$$g\left(x, \sum_{k\ge 0} \varepsilon^k h_k, t, \varepsilon\right) = B(x,t) \sum_{k\ge 1} \varepsilon^k h_k + \sum_{k \ge 1} \varepsilon^k g^{(k)}(x, h_0, \ldots, h_{k-1}, t),$$

where the matrix $B(x,t) \equiv (\partial g/\partial y)(x,h_0,t,0)$, and where

$g(x,h^{(0)}(x,t),t,0) = 0.$

Substituting these formal expansions in (33) and equating like powers of ε, we obtain

$$\frac{\partial h_{k-1}}{\partial t} + \sum_{0\le p\le k-1} \frac{\partial h_p}{\partial x} f^{(k-1-p)} = Bh_k + g^{(k)}.$$

Since B is invertible

$$h_k = B^{-1}\left[g^{(k)} - \frac{\partial h_{k-1}}{\partial t} - \sum_{0\le p\le k-1} \frac{\partial h_p}{\partial x} f^{(k-1-p)}\right].$$

A.3 Stability of slow integral manifolds

In applications it is often assumed that the spectrum of the Jacobian matrix

$g_y(x,h(x,t),t,0)$

is located in the left half plane. Under this additional hypothesis the manifold $\mathcal{M}_\varepsilon$ is exponentially attracting for $\varepsilon \in I_1$. In this case, the solution $x = x(t,\varepsilon)$, $y = y(t,\varepsilon)$ of the original system that satisfied the initial condition $x(t_0,\varepsilon) = x^0$, $y(t_0,\varepsilon) = y^0$ can be represented as

$$x(t,\varepsilon) = v(t,\varepsilon) + \varepsilon\varphi_1(t,\varepsilon),$$
$$y(t,\varepsilon) = \bar{y}(t,\varepsilon) + \varphi_2(t,\varepsilon). \tag{34}$$

There exists a point v^0 which is the initial value for a solution $v(t,\varepsilon)$ of the equation $\dot{v} = f(v,h(v,t,\varepsilon),t,\varepsilon)$; the functions $\varphi_1(t,\varepsilon)$, $\varphi_2(t,\varepsilon)$ are corrections that determine the degree to which trajectories passing near the manifold tend asymptotically to the corresponding trajectories on the manifold as t increases. They satisfy the following inequalities:

$$|\varphi_i(t,\varepsilon)| \le N|y^0 - h(x^0,t_0,\varepsilon)| \exp[-\beta(t-t_0)/\varepsilon], \quad i = 1,2, \quad \text{for } t \ge t_0. \tag{35}$$

From (34) and (35) we obtain the following *reduction principle* for a stable integral manifold defined by a function $y = h(x,t,\varepsilon)$: a solution $x = x(t,\varepsilon)$, $y = h(x(t,\varepsilon),t,\varepsilon)$ of the original system (30) is stable (asymptotically stable, unstable) if and only if the corresponding solution of the system of equations $\dot{v} = F(v,t,\varepsilon) = f(x,h(x,t,\varepsilon),t,\varepsilon)$ on the integral manifold is stable (asymptotically stable, unstable). The Lyapunov reduction principle was extended to ordinary differential systems with Lipschitz right-hand sides by Pliss (1964), and to singularly perturbed systems by Strygin and Sobolev (1977). Thanks to the reduction principle and the representation (34), the qualitative behavior of trajectories of the original system near the integral manifold may be investigated by analyzing the equation on the manifold.

The corresponding results for a class of semi-linear singularly perturbed differential systems can be found in [22].

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